CHAPTER 3

Mathematical Models and Word Problems

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.

—George Polya

Scientists and mathematicians work together to learn about, and hopefully to solve, serious problems facing all of us. One of these problems is global warming. The Earth's surface is about one degree Fahrenheit warmer than it was just 100 years ago. The presence in our atmosphere of increasing amounts of so-called greenhouse gases like methane and carbon dioxide is largely responsible.

If human activities like burning fossil fuels add methane to our air, how do scientists determine what proportion of the atmosphere is methane? One figure indicates that the concentration of methane has increased by 150% in the last 250 years. What does this figure mean? Problems dealing with the http://www.epa.gov

concentrations of substances in combination are called **mixture problems**, and they are one of the several types of word problems that this chapter addresses. The chapter project will look at some more examples related to global warming, and you can learn more at an informative government website, www.epa.gov.

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When engineers want to test a new idea, they build a model upon which to experiment. A properly constructed model is expected to behave in the same manner as the final product. In the same way, a physicist may build a model that facilitates exploration of the behavior of a natural phenomenon. Similarly, a biologist may build a model to explain the interaction between body mechanisms.

Mathematicians also build models, but they do so on paper. A **mathematical model** consists of mathematical expressions and equations that are an abstract representation of the problem. The steps in the process consist of

(a) determining the variables
(b) creating the model
(c) using the model to find a solution or solutions
(d) verifying that the solution satisfies the original problem.

The steps in the process of mathematical modeling should sound familiar. They are precisely the steps we used in Chapter 2 when we introduced you to the intriguing world of word problems. You have seen that the challenge in solving word problems lies in translating from words into mathematical equivalents, that is, in building the mathematical model.

We are now going to explore a variety of word problems in which we will show you how to build a model in an organized manner that will lead to the appropriate algebraic expressions and equations. In short, *we are going to demonstrate a method for solving word problems that is virtually foolproof.*

We pointed out in Section 2.2 that sometimes you must use a formula in order to solve a problem. This need not frighten you. For example, you already know one of the formulas we will use:

$$\text{distance} = \text{rate} \times \text{time}$$

Often, these formulas express a relationship that you use all of the time but have never written down. For example, if you have a pocketful of change and want to know how wealthy you are, you would determine the number of coins of each type (the technical term is *denomination*) and multiply each by the number of cents in that type. The formula

$$\text{value in cents} = \text{number of coins} \times \text{number of cents in each coin}$$

explicitly states the relationship that you intuitively used.

Having successfully translated from words into algebra, you must now solve the equation that you have formulated. That's the easy part: straightforward algebraic steps will lead you to a numerical solution. The final step: always check to insure that the answer "makes sense" in the context of the problem.

3.1 Coin Problems

In building a model for coin problems, you must distinguish between the *number* of coins and the *value* of the coins. For example,

n nickels have a value of $5n$ cents
n dimes have a value of $10n$ cents
n quarters have a value of $25n$ cents

If you have 8 quarters, what is their value? You find the answer by using this relationship.

> For any denomination of coins,
>
> number of coins × number of cents in each coin = value in cents

Since each quarter has a value of 25 cents, the total value of the quarters is

$$8 \times 25 = 200 \text{ cents}$$

Example 1 A Coin Problem

A purse contains $3.20 in quarters and dimes. If there are 3 more quarters than dimes, how many coins of each type are there?

Solution

In our model, we may let the unknown represent the number of quarters or the number of dimes. We make a choice. Let

$$n = \text{number of quarters}$$

then

$$n - 3 = \text{numbers of dimes}$$

since "there are 3 more quarters than dimes."

We can begin to build our model by gathering the data in the form of a chart, using the relationship

> value in cents = number of coins × number of cents in each coin

to guide us.

	Number of coins	×	Number of cents in each coin	=	Value in cents
Quarters	n		25		$25n$
Dimes	$n - 3$		10		$10(n - 3)$
Total					320

In our problem, we are told that

$$\text{total value} = (\text{value of quarters}) + (\text{value of dimes})$$

Substituting from the chart and solving,

$$\begin{aligned} 320 &= 25n + 10(n - 3) \\ 320 &= 25n + 10n - 30 \\ 350 &= 35n \\ 10 &= n \end{aligned}$$

Then

$$\begin{aligned} n &= \text{number of quarters} = 10 \\ n - 3 &= \text{number of dimes} = 7 \end{aligned}$$

Now verify that the value is \$3.20.

✔ Progress Check 1

(a) Solve Example 1, letting the unknown n represent the number of dimes.

(b) A class collected \$3.90 in nickels and dimes. If there were 6 more nickels than dimes, how many coins were there of each type?

Answers

(a) 10 quarters, 7 dimes (b) 24 dimes, 30 nickels

Example 2 A Coin Problem

A jar contains 25 coins worth \$3.05. If the jar contains only nickels and quarters, how many coins are there of each type?

Solution

We'll choose a variable to represent the number of nickels:

$$n = \text{number of nickels}$$

Can our model represent the number of quarters in terms of n? Since there is a total of 25 coins, we must have

$$25 - n = \text{number of quarters}$$

The model can then be built in the form of a chart.

	Number of coins	×	Number of cents in each coin	=	Value in cents
Nickels	n		5		$5n$
Quarters	$25 - n$		25		$25(25 - n)$
Total					305

We know that

$$\begin{aligned} \text{total value} &= (\text{value of nickels}) + (\text{value of quarters}) \\ 305 &= 5n + 25(25 - n) \\ 305 &= 5n + 625 - 25n \\ -320 &= -20n \\ n &= 16 = \text{number of nickels} \\ 25 - n &= 9 = \text{number of quarters} \end{aligned}$$

Verify that the coins have a total value of \$3.05.

✔ Progress Check 2

A pile of coins worth \$10 consisting of quarters and half-dollars is lying on a desk. If there are twice as many quarters as half-dollars, how many half-dollars are there?

Answer

10

Example 3 A Disguised Coin Problem

A man purchased 10-cent, 15-cent, and 20-cent stamps with a total value of \$8.40. If the number of 15-cent stamps is 8 more than the number of 10-cent stamps and there are 10 more of the 20-cent stamps than of the 15-cent stamps, how many of each did he receive?

Solution

This problem points out two things: (a) it is possible to phrase coin problems in terms of stamps or other objects, and (b) a “wordy” problem can be attacked by the same type of analysis.

	Number of stamps	×	Denomination of each stamp	=	Value in cents
10-cent	$n - 8$		10		$10(n - 8)$
15-cent	n		15		$15n$
20-cent	$n + 10$		20		$20(n + 10)$
Total					840

We let n be the number of 15-cent stamps (since the 10-cent and 20-cent stamps are specified in terms of the 15-cent stamps). Since

$$\text{total value} = \begin{pmatrix}\text{value of} \\ \text{10-cent stamps}\end{pmatrix} + \begin{pmatrix}\text{value of} \\ \text{15-cent stamps}\end{pmatrix} + \begin{pmatrix}\text{value of} \\ \text{20-cent stamps}\end{pmatrix}$$

we have

$$840 = 10(n - 8) + 15n + 20(n + 10)$$
$$840 = 10n - 80 + 15n + 20n + 200$$
$$840 = 45n + 120$$
$$720 = 45n$$
$$16 = n$$

Thus,

$$n = \text{number of 15-cent stamps} = 16$$
$$n - 8 = \text{number of 10-cent stamps} = 8$$
$$n + 10 = \text{number of 20-cent stamps} = 26$$

Verify that the total value is $8.40.

✔ Progress Check 3

The pretzel vendor finds that her coin-changer contains $8.75 in nickels, dimes, and quarters. If there are twice as many dimes as nickels and 10 fewer quarters than dimes, how many of each kind of coin are there?

Answer

15 nickels, 30 dimes, and 20 quarters

Exercise Set 3.1

1. A soda machine contains $3.00 in nickels and dimes. If the number of dimes is 5 times more than twice the number of nickels, how many coins of each type are there?
2. A donation box has $8.50 in nickels, dimes, and quarters. If there are twice as many dimes as nickels, and 4 more quarters than dimes, how many coins of each type are there?
3. A wallet has $460 in $5, $10, and $20 bills. The number of $5 bills exceeds twice the number of $10 bills by 4, while the number of $20 bills is 6 fewer than the number of $10 bills. How many bills of each type are there?
4. A traveler buys $990 in traveler's checks, in $10, $20, and $50 denominations. The number of $20 checks is 3 less than twice the number of $10 checks, while the number of $50 checks is 5 less than the number of $10 checks. How many traveler's checks were bought in each denomination?
5. A movie theater charges $5 admission for an adult and $3 for a child. If 700 tickets were sold and the total revenue received was $2900, how many tickets of each type were sold?
6. At a gambling casino a red chip is worth $5, a green one $2, and a blue one $1. A gambler buys $27 worth of chips. The number of green chips is 2 more than 3 times the number of red ones, while the number of blue chips is 3 less than twice the number of red ones. How many chips of each type did the gambler get?
7. A student buys 5-cent, 10-cent, and 15-cent stamps, with a total value of $6.70. If the number of 5-cent stamps is 2 more than the number of 10-cent stamps, while the number of 15-cent stamps is 5 more than one half the number of 10-cent stamps, how many stamps of each denomination did the student obtain?
8. A railroad car, designed to carry containerized cargo, handles crates that weigh 1, $\frac{1}{2}$, and $\frac{1}{4}$ ton. On a certain day, the railroad car carries 17 tons of cargo. If the number of $\frac{1}{2}$-ton containers is twice the number of 1-ton containers, while the number of $\frac{1}{4}$-ton containers

is 8 more than 4 times the number of 1-ton containers, how many containers of each type are in the car?

9. An amateur theater group is converting a large classroom into an auditorium for a forthcoming play. The group will sell \$3, \$5, and \$6 tickets. They want to receive exactly \$503 from the sale of the tickets. If the number of \$5 tickets to be sold is twice the number of \$6 tickets, and the number of \$3 tickets is 1 more than 3 times the number of \$6 tickets, how many tickets of each type are there?

10. An amusement park sells 25-cent, 50-cent, and \$1 tickets and a teacher purchases \$32.50 worth of tickets. A student remarks that there are twice as many 50-cent tickets as there are \$1 tickets and that the number of 25-cent tickets is 30 more than the number of 50-cent tickets. How many tickets of each type are there?

11. During its annual picnic, a company supplies lemonade for all employees and their families. The picnic committee has purchased twice as many pint jugs as quart jugs and 8 fewer gallon jugs than quart jugs. How many jugs of each type are there if 22 gallons of lemonade were purchased? (*Hint*: There are 2 pints to a quart and 4 quarts to a gallon.)

12. A gym offers a variety of weights for use by its members. If there are 6 more 50-pound weights than 100-pound weights and three times as many 20-pound weights as 50-pound weights, for a total of 3180 pounds, how many of each weight are there?

3.2 Investment Problems

The class of investment problems that we are going to solve involves simple interest. As an example, assume that you invest \$500 (called the **principal**) at an annual interest rate of 6%. Then the interest I available at year's end is

$$I = (0.06)(500) = 30$$

In this example, you have earned \$30 in interest. We can generalize and develop a formula that will form the basis for our modeling of these investment problems.

$$\text{simple annual interest} = \text{principal} \times \text{annual rate}$$

or

$$I = P \cdot r$$

This formula will be used in all investment problems.

Example 1 Investing at Simple Interest

A part of \$7000 is invested at 6% annual interest and the remainder at 8%. If the total amount of annual interest is \$460, how much was invested at each rate?

Solution

Let

$$n = \text{amount invested at } 6\%$$

then

$$7000 - n = \text{amount invested at } 8\%$$

since the total amount is $7000. The model can then be built in the form of a chart.

	Principal	×	Rate	=	Interest
6% portion	n		0.06		$0.06n$
8% portion	$7000 - n$		0.08		$0.08(7000 - n)$
Total					460

Since the total interest is the sum of the interest from the two parts,

$$460 = 0.06n + 0.08(7000 - n)$$
$$460 = 0.06n + 560 - 0.08n$$
$$0.02n = 100$$
$$n = \$5000 = \text{portion invested at } 6\%$$
$$7000 - n = \$2000 = \text{portion invested at } 8\%$$

✔ Progress Check 1

A club decides to invest a part of $4600 in stocks earning 4.5% annual dividends, and the remainder in bonds paying 7.5%. How much must the club invest in each to obtain a net return of 5.4%?

Answer

$3220 in stocks, $1380 in bonds

Example 2 Investing at Simple Interest

A part of $12,000 is invested at 5% annual interest, and the remainder at 9%. The annual income on the 9% investment is $100 more than the annual income on the 5% investment. How much is invested at each rate?

Solution

Let

$$n = \text{amount invested at } 5\%$$

then

$$12{,}000 - n = \text{amount invested at } 9\%$$

We can then model the information in the form of a chart.

	Principal	×	Rate	=	Interest
5% investment	n		0.05		$0.05n$
9% investment	$12{,}000 - n$		0.09		$0.09(12{,}000 - n)$

Since the interest on the 9% investment is $100 more than the interest on the 5% investment,

$$\begin{aligned} 0.09(12{,}000 - n) &= 0.05n + 100 \\ 1080 - 0.09n &= 0.05n + 100 \\ 980 &= 0.14n \\ n &= 7000 \end{aligned}$$

Thus, $7000 is invested at 5% and $5000 at 9%.

✔ Progress Check 2

$7500 is invested in two parts yielding 5% and 15% annual interest. If the interest earned on the 15% investment is twice that earned on the 5% investment, how much is invested in each?

Answer

$4500 at 5%, $3000 at 15%

Example 3 An Inventory Investment Problem

A shoe store owner had $6000 invested in inventory. The profit on women's shoes was 35%, while the profit on men's shoes was 25%. If the profit on the entire stock was 28%, how much was invested in each type of shoe?

Solution

Let

$$n = \text{amount invested in women's shoes}$$

then

$$6000 - n = \text{amount invested in men's shoes}$$

In chart form, the model now looks like this:

	Principal	×	Rate	=	Profit
Women's shoes	n		0.35		$0.035n$
Men's shoes	$6000 - n$		0.25		$0.25(6000 - n)$
Total stock	6000		0.28		0.28(6000)

The profit on the entire stock was equal to the sum of the profits on each portion:

$$\begin{aligned} 0.28(6000) &= 0.35n + 0.25(6000 - n) \\ 1680 &= 0.35n + 1500 - 0.25n \\ 180 &= 0.1n \\ n &= 1800 \end{aligned}$$

The store owner had invested $1800 in women's shoes and $4200 in men's shoes.

✔ Progress Check 3

An automobile dealer has $55,000 invested in compacts and midsize cars. The profit on sales of the compacts is 10%, and the profit on sales of midsize cars is 16%. How much did the dealer invest in compact cars if the overall profit on the total investment is 12%?

Answer

$36,666.67

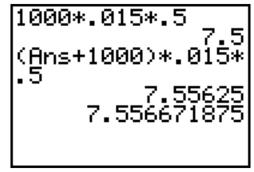
Exercise Set 3.2

1. A part of $8000 was invested at 7% annual interest, and the remainder at 8%. If the total annual interest is $590, how much was invested at each rate?
2. A $20,000 scholarship endowment fund is to be invested in two ways: part in a stock paying 5.5% annual interest in dividends and the remainder in a bond paying 7.5%. How much should be invested in each to obtain a net yield of 6.8%?
3. To help pay for his child's college education, a father invests $10,000 in two separate investments: part in a certificate of deposit paying 8.5% annual interest, the rest in a mutual fund paying 7%. The annual income on the certificate of deposit is $200 more than the annual income on the mutual fund. How much is invested in each type of investment?
4. A bicycle store selling 3-speed and 10-speed models has $16,000 in inventory. The profit on a 3-speed is 11%, while the profit on a 10-speed model is 22%. If the profit on the entire stock is 19%, how much was invested in each type of bicycle?
5. A film shop carrying black-and-white film and color film has $4000 in inventory. The profit on black-and-white film is 12%, and the profit on color film is 21%. If the annual profit on color film is $150 less than the annual profit on black-and-white film, how much was invested in each type of film?
6. A widow invested one third of her assets in a certificate of deposit paying 6% annual interest, one sixth of her assets in a mutual fund paying 8%, and the remainder in a stock paying 8.5%. If her total annual income from these investments is $910, what was the total amount invested by the widow?
7. A trust fund has invested $8000 at 6% annual interest. How much additional money should be invested at 8.5% to obtain a return of 8% on the total amount invested?
8. A businessman invested a total of $12,000 in two ventures. In one he made a profit of 8% and in the other he lost 4%. If his net profit for the year was $120, how much did he invest in each venture?
9. A retiree invested a certain amount of money at 6% annual interest; a second amount, which is $300 more than the first amount, at 8%; and a third amount, which is 4 times as much as the first amount, at 10%. If the total annual income from these investments is $1860, how much was invested at each rate?
10. A finance company lends a certain amount of money to Firm A at 7% annual interest; an amount $100 less than that lent to Firm A is lent to Firm B at 8%; and an amount $200 more than that lent to Firm A is lent to Firm C at 8.5%. If the total annual income is $126.50, how much was lent to each firm?
11. A prospective bridegroom wants to buy an engagement ring. Two jewelry stores each show him a ring at a cost of $2400. One jeweler requires a 20% down payment with the balance to be paid at the end of one year at 11% simple interest. The other jeweler requires a 25% down payment with the balance to be paid at the end of one year at 12% simple interest. What is the difference in total cost?
12. Because payment is one month overdue, a customer receives a department store bill for $332.92 that includes a 1.5% interest charge for late payment. What was the original amount of the bill?

13. An art dealer is ready to sell a Goya drawing and a Monet watercolor for which he paid a total of $45,000. If the Goya appreciated 83% and the Monet appreciated 72%, how much profit will he realize on each if he is offered $80,700 for both?

14. A small firm borrows $1000 from a stockholder at a simple interest rate of 7.5%. The company secretary lends the firm an additional sum at a simple interest rate of 8.25%. At the end of one year, the firm repays a total of $3997.75. How much did the secretary lend the firm and what is the simple interest rate on the total loan?

15. Use your graphing calculator to investigate what happens if you calculate interest more often than once per year (this is called *compound interest*). Suppose $1000 is invested at an interest rate of 1.5 % and the interest is computed after 6 months; then this interest is added to the principal so that it may earn interest for the next 6 months. (See below.)



The total interest for the year is $7.50 plus $7.56, or $15.06, rather than the $15.00 we would expect from simple interest.

Repeat the procedure above for quarterly (four times per year) and monthly compounding. Put your findings in a table.

3.3 Distance (Uniform Motion) Problems

Here is the basic formula for solving distance problems:

$$\text{distance} = \text{rate} \times \text{time}$$
or
$$d = r \cdot t$$

For instance, an automobile traveling at an average speed of 50 miles per hour for 3 hours will travel a distance of

$$\begin{aligned} d &= r \cdot t \\ &= 50 \cdot 3 = 150 \text{ miles} \end{aligned}$$

The relationships that permit you to write an equation are sometimes obscured by the words. Here are some questions to ask as you set up a distance problem:

(a) Are there two distances that are equal? Will two objects have traveled the same distance? Is the distance on a return trip the same as the distance going?

(b) Is the sum (or difference) of two distances equal to a constant? When two objects are traveling toward each other, they meet when the sum of the distances traveled by each equals the original distance between them.

Example 1 A Distance Problem

Two trains leave New York for Chicago. The first train travels at an average speed of 60 miles per hour, while the second train, which departs an hour later, travels at an average speed of 80 miles per hour. How long will it take the second train to overtake the first train?

Solution

Since we are interested in the time the second train travels, we choose to let

$$t = \text{number of hours second train travels}$$

then

$$t + 1 = \text{number of hours first train travels}$$

since the first train departs one hour earlier. In chart form, the model now looks like this:

	Rate	×	Time	=	Distance
First train	60		$t + 1$		$60(t + 1)$
Second train	80		t		$80t$

At the moment the second train overtakes the first, they must both have traveled the *same* distance.

$$\begin{aligned} 60(t + 1) &= 80t \\ 60t + 60 &= 80t \\ 60 &= 20t \\ 3 &= t \end{aligned}$$

It will take the second train 3 hours to catch up with the first train.

✔ Progress Check 1

A light plane leaves the airport at 9 A.M. traveling at an average speed of 200 miles per hour. At 11 A.M. a jet plane departs and follows the same route. If the jet travels at an average speed of 600 miles per hour, at what time will the jet overtake the light plane?

Answer

12 noon

Warning

The units of measurement of rate, time, and distance must be consistent. If a car travels at an average speed of 40 miles per hour for 15 minutes, then the distance covered is

$$d = r \cdot t$$

$$d = 40 \cdot \frac{1}{4} = 10 \text{ miles}$$

since 15 minutes = $\frac{1}{4}$ hour.

Example 2 A Distance Problem

A jogger running at the rate of 4 miles per hour takes 45 minutes more than a car traveling at 40 miles per hour to cover a certain course. How long does it take the jogger to complete the course and what is the length of the course?

Solution

Notice that time is expressed in both minutes and hours. Let's choose hours as the unit of time and let

$$t = \text{time for the jogger to complete the course}$$

then

$$t - \frac{3}{4} = \text{time for the car to complete the course}$$

since the car takes 45 minutes (= $\frac{3}{4}$ hour) less time. We can then model the information in the form of a chart.

	Rate	×	Time	=	Distance
Jogger	4		t		$4t$
Car	40		$t - \frac{3}{4}$		$40\left(t - \frac{3}{4}\right)$

Since the jogger and car travel the same distance,

$$4t = 40\left(t - \frac{3}{4}\right) = 40t - 30$$
$$30 = 36t$$
$$\frac{5}{6} = t$$

The jogger takes $\frac{5}{6}$ hour or 50 minutes. The distance traveled is

$$4t = 4 \cdot \frac{5}{6} = \frac{20}{6} = 3\frac{1}{3} \text{ miles}$$

✔ Progress Check 2

The winning horse finished the race in 3 minutes; a losing horse took 4 minutes. If the average rate of the winning horse was 5 feet per second more than the average rate of the slower horse, find the average rates of both horses.

Answer

Winner: 20 feet per second; loser: 15 feet per second

Example 3 A Distance Problem

At 2 P.M. a plane leaves Boston for San Francisco, traveling at an average speed of 500 miles per hour. Two hours later a plane departs from San Francisco to Boston traveling at an average speed of 600 miles per hour. If the cities are 3200 miles apart, at what time do the planes pass each other?

Solution

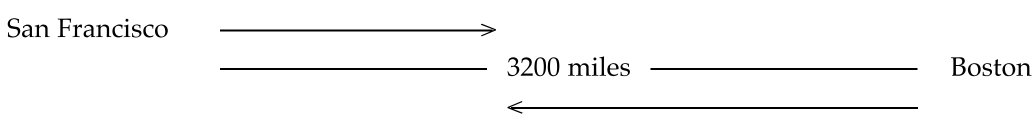
Let

$$t = \text{the number of hours after 2 P.M. at which the planes meet}$$

Let's piece together the information that we have. The model can then be built in the form of a chart.

	Rate	$\times$	Time	$=$	Distance
From Boston	500		t		$500t$
From San Francisco	600		$t-2$		$600(t-2)$

At the moment that the planes pass each other, the sum of the distances traveled by both planes must be 3200 miles.



Thus,

$$\begin{aligned} 3200 &= 500t + 600(t-2) \\ 3200 &= 500t + 600t - 1200 \\ 4400 &= 1100t \\ 4 &= t \end{aligned}$$

The planes meet 4 hours after the departure of the plane from Boston.

✔ Progress Check 3

Two cyclists start at the same time from the same place and travel in the same direction. If one cyclist averages 16 miles per hour and the second averages 20 miles per hour, how long will it take for them to be 12 miles apart?

Answer

3 hours

Exercise Set 3.3

1. Two trucks leave Philadelphia for Miami. The first truck to leave travels at an average speed of 50 kilometers per hour. The second truck, which leaves 2 hours later, travels at an average speed of 55 kilometers per hour. How long will it take the second truck to overtake the first truck?
2. Jackie either drives or bicycles from home to school. Her average speed when driving is 36 miles per hour, and her average speed when bicycling is 12 miles per hour. If it takes her $\frac{1}{2}$ hour less to drive to school than to bicycle, how long does it take to drive to school, how long does it take to bicycle to school, and how far is the school from her home?
3. Professors Roberts and Jones, who live 676 miles apart, are exchanging houses and jobs for four months. They start out for their new locations at exactly the same time, and they meet after 6.5 hours of driving. If their average speeds differ by 4 miles per hour, what is each professor's average speed?
4. Steve leaves school by moped for spring vacation. Forty minutes later his roommate, Frank, notices that Steve forgot to take his camera, so Frank decides to try to catch up with Steve by car. If Steve's average speed is 25 miles per hour and Frank averages 45 miles per hour, how long does it take Frank to overtake Steve?
5. A tour boat makes the round trip from the mainland to a fishing village in 6 hours. If the average speed of the boat going to the village is 15 miles per hour and the average speed returning is 12 miles per hour, how far from the mainland is the island?
6. Two cars start out from the same point at the same time and travel in opposite directions. If their average speeds are 36 and 44 miles per hour, respectively, after how many hours will they be 360 miles apart?
7. An express train and a local train start out from the same point at the same time and travel in opposite directions. The express train travels twice as fast as the local train. If after 4 hours they are 480 kilometers apart, what is the average speed of each train?
8. Two planes start out from the same place at the same time and travel in the same direction. One plane has an average speed of 400 miles per hour and the other plane has an average speed of 480 miles per hour. After how many hours will they be 340 miles apart?
9. Two cyclists start out at the same time from points that are 395 kilometers apart and travel toward each other. The first cyclist travels at an average speed of 40 kilometers per hour, and the second travels at an average speed of 50 kilometers per hour. After how many hours will they be 35 kilometers apart?
10. It takes a student 8 hours to drive from her home back to college, a distance of 580 kilometers. Before lunch her average speed is 80 kilometers per hour and after lunch it is 60 kilometers per hour. How many hours does she travel at each rate?

3.4 Mixture Problems

One type of mixture problem involves mixing commodities, say, two or more types of nuts, to obtain a mixture with a desired value. To form a suitable model, we will need to use a number of "common sense" relationships. If the commodities are measured in pounds, these are

number of pounds × price per pound = value of commodity

pounds in mixture = sum of pounds of each commodity

value of mixture = sum of values of individual commodities

Example 1 A Coffee Mixture

How many pounds of Brazilian coffee worth \$5 per pound must be mixed with 20 pounds of Colombian coffee worth \$4 per pound to produce a mixture worth \$4.20 per pound?

Solution

Let

$$n = \text{number of pounds of Brazilian coffee}$$

The model can then be built in the form of a chart.

Type of coffee	Number of pounds	×	Price per pound	=	Value in cents
Brazilian	n		500		$500n$
Colombian	20		400		8000
Mixture	$n + 20$		420		$420(n + 20)$

(Note that the weight of the mixture equals the sum of the weights of the Brazilian and Colombian coffees going into the mixture.) Since the value of the mixture is the sum of the values of the two types of coffee, we have

$$\begin{aligned} 420(n + 20) &= 500n + 8000 \\ 420n + 8400 &= 500n + 8000 \\ 400 &= 80n \\ 5 &= n \end{aligned}$$

We must add 5 pounds of Brazilian coffee.

✔ Progress Check 1

How many pounds of macadamia nuts worth \$4 per pound must be mixed with 4 pounds of cashews worth \$2.50 per pound and 6 pounds of pecans worth \$3 per pound to produce a mixture that is worth \$3.20 per pound?

Answer

5 pounds

Example 2 A Mixture of Chocolates

Caramels worth \$1.75 per pound are to be mixed with cream chocolates worth \$2 per pound to make a 5-pound mixture that will be sold at \$1.90 per pound. How many pounds of each are needed?

Solution

Let n = number of pounds of caramels. Displaying all of the information, we have

Type of candy	Number of pounds	×	Price per pound	=	Value in cents
Caramels	n		175		$175n$
Cream chocolates	$5 - n$		200		$200(5 - n)$
Mixture	5		190		950

(Note that the number of pounds of cream chocolates is the weight of the mixture less the weight of the caramels.) Since the value of the mixture is the sum of the values of the two components, we have

$$950 = 175n + 200(5 - n)$$
$$950 = 175n + 1000 - 200n$$
$$25n = 50$$
$$n = 2$$

We must have 2 pounds of caramels and 3 pounds of cream chocolates.

✔ Progress Check 2

How many gallons of oil worth 55¢ per gallon and how many gallons of oil worth 75¢ per gallon must be mixed to obtain 40 gallons of oil worth 60¢ per gallon?

Answer

30 gallons of the 55-cent oil and 10 gallons of the 75-cent oil

Liquid Mixtures

A second type of mixture problem involves solutions containing different concentrations of materials. For instance, a 40-gallon drum of a solution that is 75% acid contains (40)(0.75) = 30 gallons of acid. If the solutions are measured in gallons, the relationship we need is

number of gallons of solution	×	% of component A	=	number of gallons of component A

The other relationships we need are really the same as in our first type of mixture problem.

number of gallons in mixture	=	sum of the number of gallons in each solution
number of gallons of component A in mixture	=	sum of the number of gallons of component A in each solution

Example 3 A Liquid Mixture

A 40% acid solution is to be mixed with a 75% acid solution to produce 140 gallons of a solution that is 50% acid. How many gallons of each solution must be used?

Solution

For our model, we let

$$n = \text{number of gallons of the 40\% acid solution}$$

Then

$$140 - n = \text{number of gallons of the 75\% acid solution}$$

since the number of gallons in the mixture is the sum of the number of gallons in each contributing solution. We can then model the information in the form of a chart.

	Number of gallons	×	% acid	=	Number of gallons of acid
40% solution	n		40		$0.40n$
75% solution	$140 - n$		75		$0.75(140 - n)$
Mixture	140		50		70

Since the number of gallons of acid in the mixture is the sum of the number of gallons of acid in each solution (see Figure 1), we have

$$\begin{aligned} 70 &= 0.40n + 0.75(140 - n) \\ 70 &= 0.40n + 105 - 0.75n \\ -35 &= -0.35n \\ n &= 100 \text{ gallons} \\ 140 - n &= 40 \text{ gallons} \end{aligned}$$

Thus, we mix 100 gallons of the 40% solution with 40 gallons of the 75% solution to produce 140 gallons of the 50% solution.

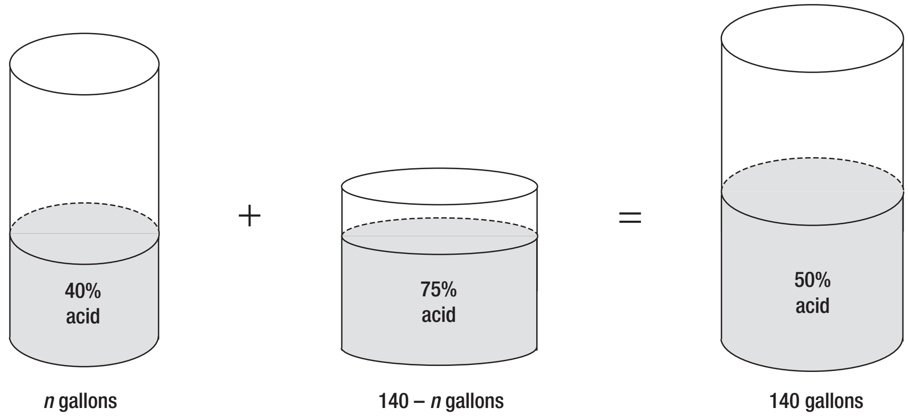


Figure 1 **Concentration of Acid**

✔ Progress Check 3

How many gallons of milk that is 22% butterfat must be mixed with how many gallons of cream that is 60% butterfat to produce 19 gallons of a mixture that is 40% butterfat?

Answer

10 gallons of milk and 9 gallons of cream

Example 4 A Mixture of Alloys

How many ounces of an alloy that is 30% tin must be mixed with 15 ounces of an alloy that is 12% tin to produce an alloy that is 24% tin?

Solution

Let n = number of ounces of the 30% tin alloy. The alloy may be treated as a solution, and we can display the information for our model.

	Number of ounces	×	% tin	=	Number of ounces of tin
30% alloy	n		30		$0.30n$
12% alloy	15		12		1.8
Mixture	$n + 15$		24		$0.24(n + 15)$

(Note that the number of ounces in the mixture is the sum of the number of ounces in the alloys going into the mixture.) Since the number of ounces of *tin* in the mixture is the sum of the number of ounces of tin in each alloy, we have

$$0.24(n + 15) = 0.30n + 1.8$$
$$0.24n + 3.6 = 0.30n + 1.8$$
$$1.8 = 0.06n$$
$$n = 30$$

Thus, we need to add 30 ounces of the 30% alloy to 15 ounces of the 12% alloy.

✔ Progress Check 4

How many pounds of a 25% copper alloy must be added to 50 pounds of a 55% copper alloy to produce an alloy that is 45% copper?

Answer

We need to add 25 pounds of 25% copper alloy.

Example 5 Reducing a Mixture

A tank contains 40 gallons of water and 10 gallons of alcohol. How many gallons of water must be removed if the remaining solution is to be 30% alcohol?

Solution

Let n = number of gallons of water to be removed. This problem is different since we are removing water. We can model the information in the form of a chart.

	Number of gallons	×	% alcohol	=	Gallons of alcohol
Original solution	50		20		10
Water removed	n		0		0
New solution	$50 - n$		30		$0.3(50 - n)$

(Note that the water removed has 0% alcohol!) The number of gallons of alcohol in the new solution is the same amount as in the original solution, since only water has been removed.

$$0.3(50 - n) = 10$$
$$15 - 0.3n = 10$$
$$5 = 0.3n$$
$$n = 16\frac{2}{3}$$

Thus, we must remove $16\frac{2}{3}$ gallons of water.

✔ Progress Check 5

A tank contains 90 quarts of an antifreeze solution that is 50% antifreeze. How much water should be removed to raise the antifreeze level to 60% in the new solution?

Answer

15 quarts of water should be removed.

Exercise Set 3.4

1. How many pounds of raisins worth \$1.50 per pound must be mixed with 10 pounds of peanuts worth \$1.20 per pound to produce a raisin-peanut mixture worth \$1.40 per pound?
2. How many ounces of Ceylon tea worth \$1.50 per ounce and how many ounces of Formosa tea worth \$2.00 per ounce must be mixed to obtain a mixture of 8 ounces that is worth \$1.85 per ounce?
3. A copper alloy that is 40% copper is to be combined with a copper alloy that is 80% copper to produce 120 kilograms of an alloy that is 70% copper. How many kilograms of each alloy must be used?
4. How many liters of an ammonia solution that is 20% ammonia must be mixed with 20 liters of an ammonia solution that is 48% ammonia to produce a solution that is 36% ammonia?
5. A vat contains 60 gallons of a 15% saline solution. How many gallons of water must be evaporated so that the resulting solution will be 20% saline?
6. How many grams of pure silver must be added to 30 grams of an alloy that is 50% silver to obtain an alloy that is 60% silver?
7. How much water must be added to dilute 10 quarts of a solution that is 18% iodine so that the resulting solution will be 12% iodine?
8. A vat contains 27 gallons of water and 9 gallons of acetic acid. How many gallons of water must be evaporated if the remaining solution is to be 40% acetic acid?
9. How many pounds of a fertilizer worth \$3 per pound must be combined with 12 pounds of a weed killer worth \$6 per pound and 18 pounds of phosphate worth \$6 per pound to produce a mixture worth \$4.80 per pound?
10. A producer of packaged frozen vegetables wants to market the product at \$1.20 per kilogram. How many kilograms of green beans worth \$1 per kilogram must be mixed with 100 kilograms of corn worth \$1.30 per kilogram and 90 kilograms of peas worth \$1.40 per kilogram to produce the required mixture?

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Key Ideas for Review

Topic	Page	Key Idea
Mathematical model	89	When you convert a word problem into mathematical expressions, equations, and inequalities, you create a mathematical model for the problem.
Formulas associated with word problems		
coin problems	90	number of coins × number of cents in each coin = value in cents
investment problems	94	principal × rate = interest
distance (uniform motion) problems	98	rate × time = distance
mixture problems	102	number of pounds × price per pound = value amount of solution × % of component A = amount of component A

Review Exercises

Solutions to exercises whose numbers are in bold are in the Solutions section in the back of the book.

3.1

1. A church collection box contains $5.35 in dimes, quarters, and half-dollars. If the number of dimes is twice the number of quarters, and the number of half-dollars is one less than three times the number of quarters, how many coins of each denomination are there?

2. A certain electronic device consists of 16-transistor, 48-transistor, and 64-transistor components. The number of 48-transistor components is two less than the number of 16-transistor components, and the number of 64-transistor components is three less than twice the number of 16-transistor components. If the device contains a total of 480 transistors, how many components of each type are required?

3. A freighter carries a load of 2-ton, 5-ton, and 8-ton slabs of steel. The number of 5-ton slabs is 20 fewer than the number of 2-ton slabs and the number of 8-ton slabs is one more than twice the number of 2-ton slabs. If the load being carried weighs 575 tons, how many slabs of each type are being carried?

4. Suppose that you receive a package with a total postage of $5.60 that is made up of 20-cent, 40-cent, and 1-dollar stamps. If the number of 40-cent stamps is three more than the number of 20-cent stamps, and the number of 1-dollar stamps is two fewer than the number of 20-cent stamps, how many stamps of each denomination are there on the package?

3.2

5. Part of a lump-sum death payment of $40,000 was invested at 10% annual interest, and the rest was invested at 8% annual interest. If the total annual interest is $3500, how much was invested at each rate?

6. A record shop selling classical and popular music has $12,000 worth of music in inventory. The profit on classical music is 15%, while the profit on popular music is 20%. If the annual profit on classical music is $1000 less than the annual profit on popular music, how much inventory does the shop carry in each type of music? (Assume that the entire inventory is sold.)

7. An investment club invested a total of $7000 in two real estate limited partnerships. In one

partnership they make a profit of 10% for the year and in the other they have a loss of 5% for the year. If the net annual profit is \$325, how much was invested in each limited partnership?

8. A finance company lent a certain amount of money to the AB company at 8% annual interest. An amount \$500 more than that lent to the AB company was lent to the CD company at 10% annual interest, and an amount \$400 less than the amount lent to the AB company was lent to the EF company at 12% annual interest. If the total annual interest received by the finance company is \$1502, how much was lent to each borrower?

3.3 **9.** Two aircraft start from the same point at the same time flying in opposite directions. The faster aircraft travels twice as fast as the other one. After 5 hours of travel they are 1500 miles apart. Find the average speed of each aircraft.

10. Two joggers start to run toward each other at the same time from points that are 12 miles apart, at average speeds of 10 miles per hour and 9 miles per hour, respectively. After how many hours will they be 0.6 mile apart?

11. Two buses, traveling at average speeds of 50 and 55 miles per hour, respectively, leave Los Angeles for Chicago at the same time. After how many hours are they 60 miles apart?

12. Two airplanes leave at the same time from points 3150 miles apart, traveling toward each other, and they pass each other after 3.5 hours of flying. If their average speeds differ by 100 miles per hour, what is the average speed of each airplane?

3.4 13. How many pounds of cashews worth \$4.00 per pound must be mixed with 6 pounds of walnuts worth \$2.00 per pound to yield a mixture worth \$2.50 per pound?

14. A vat contains 100 gallons of a 20% potassium solution. How many gallons of water must be evaporated to get a 25% solution?

15. How many pounds of ground beef that is 25% fat must be blended with 10 pounds of ground veal that is 10% fat to produce a mixture that is 15% fat?

16. How many pounds of Colombian coffee worth \$4.00 per pound must be mixed with how many pounds of Jamaican coffee worth \$5.00 per pound to produce 25 pounds of a mixture that will be sold at \$4.80 per pound?

17. Atmospheric concentrations are usually measured in parts per billion, by volume (ppbv). So, a concentration of carbon dioxide equal to 13 ppbv means that, for every billion volume units (say, milliliters) of the atmosphere, there are 13 units of carbon dioxide.

Convert an atmospheric concentration of carbon dioxide of 13 ppbv to a percent (that is, what percent of this atmosphere would be carbon dioxide?)

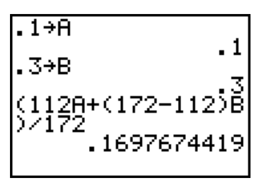
18. In 2002 the atmospheric concentration of the greenhouse gas methane was about 1751 parts per billion by volume (ppbv). This means that in a one-billion liter volume of the atmosphere, we should find about 1751 liters of methane gas.

Suppose we had a 100 L sample of gas with this concentration of methane. How much methane, in mL, would have to be added to increase the concentration to 150% of its current level? (Hint: We want to increase the concentration to 1.5×1751, or 2626.5 ppbv.)

19. Store the following values in your graphing calculator as A and B: $A = 0.1$, $B = 0.3$ (see the calculator screen below). Then, enter the expression

$$\frac{112A + (172 - 112)B}{172}$$

Suppose the 112 in the above expression represents liters of CO_2, and the values of A and B represent 10% and 30%, respectively.



a. What might the 172 represent? What about the 0.17?

b. Experiment with changing the values of A and B. How can this expression help you solve certain mixture problems?

Progress Test 3A

1. Translate into algebra: "The number of chairs is 3 less than 4 times the number of tables."
2. Steve is presently 6 years younger than Lisa. If the sum of their ages is 40, how old is each?
3. The width of a rectangle is 4 cm less than twice its length. If the perimeter is 12 cm, find the dimensions of the rectangle.
4. A donation box contains 30 coins consisting of nickels, dimes, and quarters. The number of dimes is 4 more than twice the number of quarters. If the total value of the coins is $2.60, how many coins of each type are there?
5. A fruit grower ships crates of oranges that weigh 30, 50, and 60 pounds each. A certain shipment weighs 1140 pounds. If the number of 30-pound crates is 3 more than one half the number of 50-pound crates, and the number of 60-pound crates is 1 less than twice the number of 50-pound crates, how many crates of each type are there?
6. A college fund has invested $12,000 at 7% annual interest. How much additional money must be invested at 9% to obtain a return of 7.8% on the total amount invested?
7. A businessperson invested a certain amount of money at 6.5% annual interest; a second amount, which is $200 more than the first amount, at 7.5%; and a third amount, which is $300 more than twice the first amount, at 9%. If the total annual income from these investments is $1962, how much was invested at each rate?
8. A moped and a car leave from the same point at the same time and travel in opposite directions. The car travels 3 times as fast as the moped. If after 5 hours they are 300 miles apart, what is the average speed of each vehicle?
9. A bush pilot in Australia picks up mail at a remote village and returns to home base in 4 hours. If the average speed going is 150 miles per hour and the average speed returning is 100 miles per hour, how far from the home base is the village?
10. An alloy that is 60% silver is to be combined with an alloy that is 80% silver to produce 120 ounces of an alloy that is 75% silver. How many ounces of each alloy must be used?
11. A beaker contains 150 cubic centimeters of a solution that is 30% acid. How much water must be evaporated so that the resulting solution will be 40% acid?

Progress Test 3B

1. Translate into algebra: "The number of Democrats is 4 more than one third the number of Republicans."
2. Separate 48 into two parts so that the larger part plus 3 times the smaller is 80.
3. One side of a triangle is 2 cm shorter than the third side, while the second side is 3 cm longer than one half the third side. If the perimeter is 15 cm, find the length of each side.
4. An envelope contains 20 discount coupons in $1, $5, and $10 denominations. The number of $5 coupons is twice the number of $10 coupons. If the total value of the coupons is $54, how many coupons of each type are there?
5. A cheese sampler with a total weight of 25 ounces of cheese contains 1-ounce, 2-ounce, and 3-ounce samples. If the number of 1-ounce samples is 3 more than the number of 3-ounce samples, and the number of 2-ounce samples is 1 less than twice the number of 3-ounce samples, how many samples of each weight are there?
6. Part of an $18,000 trust fund is to be invested in a stock paying 6% in dividends, and the remainder in a bond paying 7.2% annual interest. How much should be invested in each to obtain a net yield of 7%?
7. A woman invested a certain amount of money at 8% annual interest, and a second amount of money, $2000 greater than the first amount, at 6%. If the annual incomes on the two investments are equal, how much was invested at each rate?
8. Two trains start out at 10 A.M. from stations that are 1120 kilometers apart, and travel toward each other at

average speeds of 80 and 60 kilometers per hour, respectively. At what time will they pass each other?

9. Two charter buses leave New York for Los Angeles. The first one travels at an average speed of 40 miles per hour. The second one leaves 3 hours later and travels at an average speed of 50 miles per hour. How long will it take the second bus to overtake the first one?

10. How many pounds of lawn seed worth \$4.00 per pound must be mixed with 15 pounds of fertilizer worth \$3.00 per pound to produce a mixture worth \$3.20 per pound?

11. A vat contains 12 gallons of acid and 48 gallons of water. How much acid must be added to make a solution that is 40% acid?

Chapter 3 Project: Greenhouse Gas Concentrations and Global Warming

The atmosphere of the Earth is a mixture of gases. The relative proportions of the major components of the atmosphere (nitrogen and oxygen) don't change very much over time, but there are small amounts of other gases that can have a big effect on the planet's climate, and on its inhabitants (us!).

Greenhouse gases like carbon dioxide and methane trap heat that would otherwise be radiated out into space. A little of this is a good and necessary thing, but some human activities (like burning of fossil fuels) release these gases and increase the concentration of greenhouse gases. This has the world's scientists concerned that we may have too much of a good thing!

In this chapter project, you will learn about how concentrations of these gases are measured, and how to use percents to get an idea of what proportion of our atmosphere is composed of greenhouse gases. You will try to determine how much this proportion can change with the addition of greater quantities of these atmospheric components. Start by reading more at www.epa.gov, and then do Exercises 17, 18, and 19 in Section 3.4.

Create a table showing how much the composition of the atmosphere changes with the addition of different amounts of CO_2 and methane. (You might want to use scientific notation to express small percents more compactly.) Explain how you can use your calculator to make these computations more quickly. (Hint: See Exercise 19 in Section 3.4.)

Using the information at the website listed above, or other sources, make a graph of the changes in quantities of methane in our atmosphere over the past 200 years.